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Low-lying excitations in the square–triangle random tiling model

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Abstract. We consider the sub-dominant eigenstates of the transfer matrix for the square–triangle random tiling model on an infinite strip of width L. A numerical algorithm for generation of the corresponding solution of the Bethe ansatz is developed. Numerical finite-size scaling analysis of the associated eigenvalues reveals the presence of both integer and non-integer critical exponents. The *analytical* value of one of the non-integer exponents is found. It is also shown numerically that, along with the leading L^{-2} correction to the free-energy density, for some excitations there is a term proportional to $L^{-12/5}$.

1. Introduction

The square–triangle random tiling model was first solved numerically by Bethe ansatz [1] and then analytically in the thermodynamic limit [2]. The analytic solution provides the exact values of the entropy and the phason susceptibility for the model, but neither of these parameters is a universal quantity. At the same time, several intriguing features of the analytic solution imply that the the model reaches criticality if the tiling possesses six-fold rotational symmetry. In particular, the distribution of the roots of Bethe ansatz equations is singular for six-fold symmetric tilings. It is of interest to figure out whether this singularity has a physical significance. The results obtained so far suggest rather the contrary, because the free energy of the system is a regular function of the only available macroscopic parameters—the phason strains [2]. Nevertheless, one cannot exclude the presence of hidden macroscopic variables, the behaviour of which becomes critical at this point. This could be seen as a dramatic change in the long-range correlation functions of these variables. The Bethe ansatz for the eigenstates of the transfer matrix is not sufficient (except in rare cases) on its own to find the correlation functions. However, the numerical data on the distribution of the eigenvalues near the top of the spectrum of the transfer matrix provide circumstantial evidence on the macroscopic behaviour of the model. First of all, the presence of an irrelevant operator can be detected by the behaviour of the next-to-leading term in the dependence of the spacing of the sub-dominant eigenvalues on the size of the system [3]. The other method, providing more precise information on the scaling dimensions, is based on the hypothesis of conformal invariance. The case in point is the so-called finite-size scaling approach [4, 5]. It is well known that the critical exponents x_i for two-dimensional statistical models which exhibit conformal symmetry in the critical state are related to the spacings of the sub-dominant eigenvalues of the transfer matrix [6]

$$\log(\lambda_0) - \log(\lambda) = \frac{2\pi(n + x_i)}{L} \tag{1}$$

where λ and λ_0 are the eigenvalues of the excited and ground state correspondingly, n is a non-negative integer and the model is formulated on a strip of width L with periodic boundary conditions. Another important relation links the finite-size correction to the density of the free energy and the central charge of the corresponding conformal model [4, 5]

$$F_0 - F = \frac{\pi c}{6L^2}. \tag{2}$$

In the present paper we analyse numerically some of the low-lying excitations and compare the finite-size scaling data with formulae (1) and (2).

2. Creation of the low-lying excitations

Recall that the square–triangle random tiling model can be mapped on a lattice and that the individual tilings can be conveniently described in terms of the world lines of right- and left-moving particles [1, 2]. Following [2], the model is formulated on a strip of finite width $2M$ (after mapping) with periodic boundary conditions. For the states with n_- right-moving and n_+ left-moving particles the equations of the Bethe ansatz take the form

$$\mathrm{e}^{-M\phi}\xi_i^M = (-1)^{n_+-1}\prod_j(\xi_i - \psi_j) \qquad \mathrm{e}^{M\phi}\psi_j^M = (-1)^{M+n_--1}\prod_i(\xi_i - \psi_j) \tag{3}$$

where

$$\xi_i = \exp(2\mathrm{i}p_i + \phi) \qquad \psi_j = -\exp(-2\mathrm{i}q_j - \phi)$$

and p_i and q_j are the momenta of particles. The corresponding eigenvalue of the transfer matrix is given by

$$\Lambda = \exp\left(\mathrm{i}\sum_i p_i - \mathrm{i}\sum_j q_j\right).$$

Since the transfer matrix commutes with the translations along the periodic directions, the eigenvalues may be characterized by the momentum as well. In terms of Bethe ansatz (3), the momentum of an eigenstate is expressed as

$$P = \left(\sum_i p_i + \sum_j q_j\right) \bmod 2\pi.$$

The momentum is quantized in units of π/M because of the periodic boundary conditions:

$$m = \frac{M}{\pi}P \in Z. \tag{4}$$

The system of equations (3) has $M!/((M - n_+ - n_-)!n_+!n_-!)$ solutions, which satisfy the condition that all roots ξ_i and ψ_i are different. We are mainly interested in the solutions which correspond to the low-lying excitations of the system. Although the behaviour of the ground-state solution can be derived analytically in the thermodynamic limit, there is no clear way of doing so for the sub-dominant eigenstates. Nevertheless, the numerical solutions of system (3) can be found with full machine precision, allowing in its turn precise determination of the finite-size scaling effects on the spectrum of the transfer matrix. At first glance, numerical search of low-lying excitations can be performed by gentle perturbation of the roots ξ_i and ψ_j corresponding to the ground state, followed by the search of the nearest root of system (3). This approach has two important drawbacks: it does not guarantee that in the resulting solution all roots ξ_i and ψ_j will be different, neither does it provide us with a scheme of classification of the elementary excitations. This is why a different algorithm has been used to generate the solutions of (3) corresponding to the excited states of the system.

Let us add the phase factors in the equations of the Bethe ansatz above:

$$e^{-M\phi}\xi_i^M = e^{i\Phi_i^+}(-1)^{n_+-1}\prod_j(\xi_i-\psi_j) \qquad e^{M\phi}\psi_j^M = e^{i\Phi_j^-}(-1)^{M+n_--1}\prod_i(\xi_i-\psi_j). \tag{5}$$

Clearly, this system has physical meaning only if all phases Φ_i^+ and Φ_j^- are multiples of 2π. The important feature of system (5) is that a continuous change in the phases Φ_i^+ and Φ_j^- induces continuous evolution of the roots ξ_i and ψ_j for as long as no two of them coincide. Thus, any trajectory in the space of phases Φ_i^+ and Φ_j^-, connecting two points which are multiples of 2π corresponds to a transformation of one solution of system (5) to another. This fact forms the basis of the numerical algorithm for generation of the excited eigenstates of the square–triangle random tiling. In a nutshell, this algorithm consists of an adiabatic change in the phases Φ_i^+ and Φ_j^- interleaved with the search for the nearest solution of system (5). The merits of this algorithm with respect to that based on the random perturbation of the roots of (3) are that it gives completely reproducible results and allows us to generate the same excitation in systems of different size.

3. Numerical results

Before proceeding any further it is worth noting the important peculiarity of the square–triangle random tiling model. Since this model is not initially formulated on a lattice, applying the transfer matrix technique requires some sort of mapping it onto a lattice model (see [1, 2]). The form of the mapping, in its turn, is sensitive to the state of the model, in the sense that the distortion factor depends on the phason gradient. In the finite-size scaling considerations the density of the free energy should be known with at least $o(L^{-2})$ accuracy, which implies the same precision in the measurement of the distortion factor. To be on the safe side, only the excitations which do not modify the phason gradient are considered from here on. This implies that the number of particles of each sort n_+ and n_- and the parameter ϕ in (3) are kept constant. Another important restriction is due to the fact that the density of the particles for the state with 12-fold symmetry is equal to $1-\sqrt{3}/3$, i.e. an irrational number. That is, for an arbitrarily chosen M the deviation of the real density $n_\pm/M$ from the ideal value is of the order $1/M$. The corresponding phason gradient is of the same order, which gives rise to an $O(L^{-2})$ deviation in the value of the free-energy density. Fortunately, there exists a series of good rational approximations to the ideal density, based on the continuous fractions, for which $n_\pm/M = (1-\sqrt{3}/3)+O(M^{-2})$. The corresponding denominators are $M = 7, 26, 97, 362, 1351\ldots$. All numerical results below are obtained for this series of values of M and $n_\pm$.

Whether or not the model possesses conformal symmetry at the critical point, the conformal symmetry considerations only make sense if the rotational symmetry is preserved. This is not the case for the mapping onto a lattice used in [2]. Thus, equation (1) only makes sense if the true width of the strip L is used. This width, measured in the lengths of the steps of the transfer matrix along the strip, is equal to $2M/\sqrt{3}$ for the tilings with 12-fold symmetry. Thus, the parameter $n+x_i$ from (1) is related to the eigenvalues of the transfer matrix as follows:

$$n+x_i = \frac{M(\log(\lambda_0)-\log(\lambda))}{\pi\sqrt{3}}. \tag{6}$$

We start the consideration with the simplest finite-size effect: that of the dependence of the free-energy density of the ground state on the size of the system. For conformal models,

Table 1. The values of $6L^2(F_0 - F)/\pi$ for different values of M. For the conformal symmetric models these values should converge to the central charge of the model in the limit $M \to \infty$.

M	c
7	1.854 941 737 185 41(1)
26	1.980 571 089 403 0(2)
97	1.998 482 530 808(3)
362	1.999 890 306 60(5)
1351	1.999 992 122(1)
5042	1.999 999 46(2)
∞	2.000 000 00(5)

the corresponding correction is related to the central charge of the model by formula (2). It is interesting to see that, for the square–triangle random tiling, the numerical results (see table 1) are consistent with the scaling (2), with the value of the central charge $c = 2$. This observation is consistent with the recent result by de Gier and Nienhuis [7] that the square–triangle random tiling is equivalent to a degenerate case of the $O(n)$ model on the honeycomb lattice. In fact, according to Reshetikhin [8] if $n \leqslant 2$ this model is critical at zero temperature, and corresponds to an effective conformal field theory with central charge $c = 2$.

The solutions of equations (3), obtained as described above, do not all correspond to the eigenstates of the transfer matrix. In fact, the anti-symmetrization of the eigenfunction with respect to the momenta of the particles of the same sort implies that these momenta should be different. At the same time, the adiabatic change of $\pm 2\pi$ in one phase Φ_i^+ or Φ_j^- in most cases brings into existence, pairs of coinciding roots ξ_i or ψ_j. The reason is that for the most of the roots ξ_i and ψ_j the neighbouring positions are occupied. Only the roots near the edges of the spectrum have room to move. The situation is somewhat similar to that of the low-lying excitations in a Fermi liquid, the only difference being that the dispersion law depends itself on the distribution of particles on the energy levels. Pursuing the analogy with the one-dimensional Fermi liquid, one can expect that there are two types of excitation—one involving the particles in the vicinity of a Fermi point, and the other corresponding to the Umklapp process, i.e. giving rise to the transfer of a particle between Fermi points. Both types of excitation can be generated by application of the described algorithm.

Consider first the excitations which do not involve Umklapp processes. The simplest possible case consists of moving a particle at the very end of the spectrum to the nearest vacant position. One could expect that the resulting state depends on the choice of the particle to move (is it a right- or a left-moving particle?). This is indeed almost always the case, except when the excitation to the six-fold symmetric state of the model is considered. In this case, the excitations of the right- and left-moving particle give rise to the same solution of the Bethe ansatz equations! This can be seen graphically in figure 1, where the corresponding configuration is shown for the ground state of the model. This peculiarity of the six-fold symmetric states results from the fact that perturbation of the dispersion law for particles of the other sort, due to such excitation, cannot be considered as small even in the limit $M \to \infty$. The corresponding modification to the eigenvalue of the transfer matrix is given in table 2.

The correction tends to 1 as $M \to \infty$ to judge from the results of Richardson extrapolation. The momentum (4) of this excitation corresponds to $m = \pm 1$. It is notable

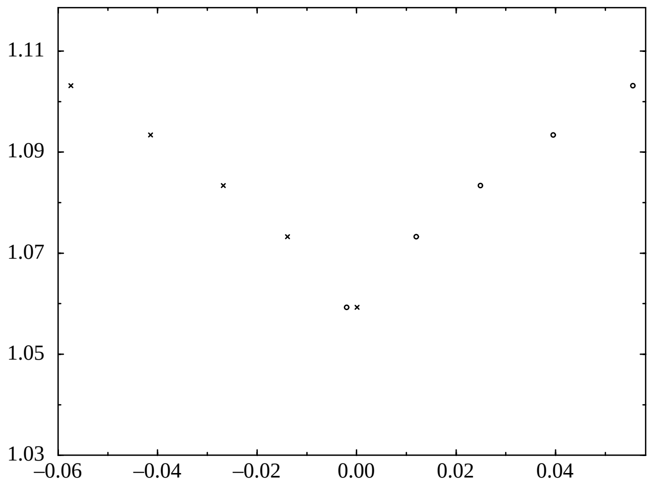

Figure 1. The configuration of roots of the Bethe ansatz for the lowest-lying particle–hole excitation near the 'Fermi point', $M = 362$. Circles and crosses represent ξ_i ψ_j correspondingly.

Table 2. The finite-size correction to the free-energy density for the lowest-lying excitation with pseudo-time reversal symmetry.

M	$L(\log(\lambda_0) - \log(\lambda))/2\pi$
26	$-0.934\,247\,009\,9499(2)$
97	$-0.979\,917\,110\,651(3)$
362	$-0.994\,426\,893\,07(5)$
1351	$-0.998\,492\,640(1)$
∞	$-1.000\,000\,021(5)$

that the singularity in the distribution of the roots of Bethe ansatz near the edge of the spectrum has apparently no effect on the next to the leading terms in (1). In fact, the closest to the integer extrapolation for $M \to \infty$ is obtained when using integer powers of M. On the other hand, taking into account that the distance between the roots of the Bethe ansatz near the end of the spectrum scales as $M^{-6/5}$, and using the qualitative arguments based on the Euler–Maclaurin integration formula, one could predict the presence of the terms, proportional to $M^{-12/5}$ in the correction to the free-energy density. As we shall see, this is indeed the case for some excitations.

To obtain more sub-dominant eigenstates, one can move the edge particles to the next vacant position. The corresponding configurations of roots for the case of one and two particles are shown on figures 2 and 3. The results for the energy of the eigenstate are given in tables 3 and 4. The values seem to converge to 2 and 3 correspondingly, but the slowness of the convergence stands out. Numerically, the closest to the integer answer is obtained when Richardson extrapolation in the powers of $M^{-2/5}$ is used (see the arguments above).

Consider now the excitations whereby a particle undergoes the Umklapp process. This would correspond to a change in one of the phases Φ_i^+ or Φ_j^- of $2\pi n_\pm$. The same result can be achieved by a simultaneous shift in all phases Φ_i^+ (or Φ_j^-) by 2π. The latter is preferable from the point of view of algorithmic efficiency. For the sake of simplicity, only

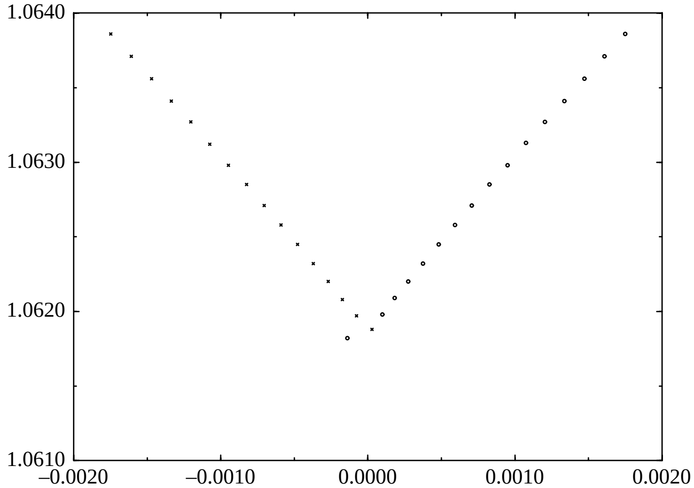

Figure 2. The configuration of roots of the Bethe ansatz for the non-symmetric particle–hole excitation, $M = 18\,817$.

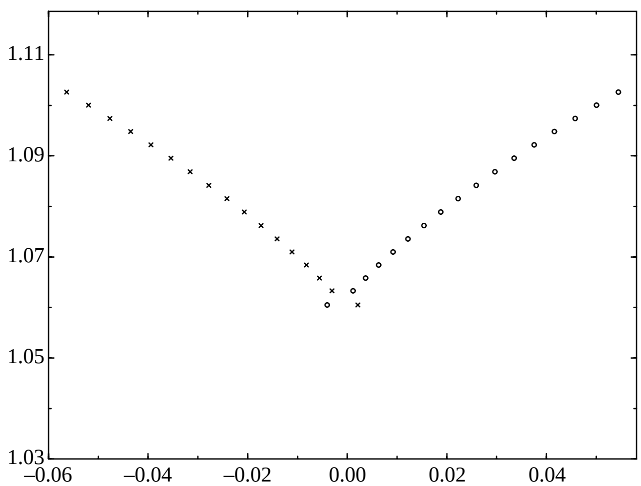

Figure 3. The configuration of roots of the Bethe ansatz for the symmetric particle–hole excitation, $M = 1351$.

the excitations which do not break the pseudo-time reversal symmetry are considered here. This implies that both the left- and right-moving particles are involved in the Umklapp process. The rescaled eigenvalues of the transfer matrix for the lowest lying excitation of this type are shown in table 5. These values are obviously converging to a non-integer number. In the next section we show how to obtain the analytic expression for it. It is notable that the values of the momentum for this eigenstate increase with the size of the system.

Table 3. The finite-size correction to the free-energy density for the particle–hole excitation without pseudo-time reversal symmetry ($m = \pm 2$).

M	$L(\log(\lambda_0) - \log(\lambda))/2\pi$	
7	−0.594 659 300 564 40(1)	+0.909 146 789 843 77(1)i
26	−1.059 131 344 659 1(2)	+0.549 869 503 473 3(2)i
97	−1.388 303 436 012(3)	+0.363 705 847 637(3)i
362	−1.618 747 777 70(5)	+0.239 684 457 31(5)i
1 351	−1.768 008 478(1)	+0.153 249 006(1)i
5 042	−1.860 708 03(2)	+0.095 328 86(2)i
18 817	−1.916 978 5(3)	+0.058 151 0(3)i
∞	−2.000 007(1)	−0.000 005(1)i

Table 4. The finite-size correction to the free-energy density for the symmetric particle–hole excitation, involving particles of two types ($m = 3$).

M	$L(\log(\lambda_0) - \log(\lambda))/2\pi$
97	−2.107 256 933 317(3)
362	−2.432 151 493 92(5)
1 351	−2.651 016 890(1)
5 042	−2.789 442 35(2)
18 817	−2.874 191 2(3)
∞	−3.000 048 0(5)

Table 5. The finite-size correction to the free-energy density and the momentum for the symmetric Umklapp excitation.

M	$L(\log(\lambda_0) - \log(\lambda))/2\pi$	m
26	−2.542 324 530 0753(2)	±4
97	−2.426 715 669 604(3)	±15
362	−2.418 993 512 67(5)	±56
1 351	−2.418 441 811(1)	±209
5 042	−2.418 402 21(2)	±780
18 817	−2.418 399 3(3)	±2911
∞	−2.418 399 1(5)	

4. Analytic solution

The excitation for which an analytic solution in the thermodynamic limit is found can be generated by simultaneous increase of all phases Φ_i^+ and Φ_j^- by a multiple of 2π. Because this perturbation does not give rise to the formation of holes in the sequences of the roots ξ_i and ψ_j of equations (3), it is natural to expect that the technique developed in [2] will still be applicable to this case. Recall that for the ground state the roots ξ_i and ψ_j of Bethe ansatz equations (3) in the limit $M \to \infty$ are concentrated along the curves Ψ and Ξ on the complex plane. This makes it possible to replace equations (3) by a system of two integral equations in the thermodynamic limit [2]:

$$f_+(\zeta) = 1/\zeta + \frac{1}{2\pi \mathrm{i}} \int_{b_{+,1}}^{b_{+,2}} \frac{f_-(z)\mathrm{d}z}{z - \zeta} \qquad f_-(\zeta) = 1/\zeta - \frac{1}{2\pi \mathrm{i}} \int_{b_{-,1}}^{b_{-,2}} \frac{f_+(z)\mathrm{d}z}{z - \zeta} \tag{7}$$

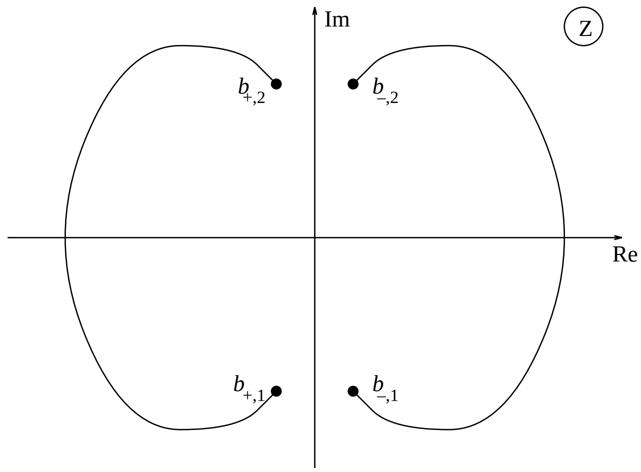

Figure 4. The integration paths for equations (7).

(the integration contours are shown on figure 4). The only degree of freedom left in (7) is that of the choice of the limits of integration $b_{+,1}$, $b_{+,2}$, $b_{-,1}$ and $b_{-,2}$. The total number of real parameters is thus equal to eight. There are three additional constraints applied to these parameters. First of all, two constraints are due to the condition that the forms $f_- \,\mathrm{d}z$ and $f_+ \,\mathrm{d}z$ take pure imaginary values on the vectors tangent to the curves Ψ and Ξ correspondingly [2]. Consequently,

$$\mathrm{Re}\left(\int_{b_{+,1}}^{b_{+,2}} f_- \,\mathrm{d}z\right) = 0 \qquad \mathrm{Re}\left(\int_{b_{-,1}}^{b_{-,2}} f_+ \,\mathrm{d}z\right) = 0. \tag{8}$$

Second, equations (7) leave the scale of the variable z undefined. The scale factor is lost in (7) during the derivation from (3), and can be restored as described in [2]. This gives rise to the additional constraint on $b_{+,1}, b_{+,2}, b_{-,1}$ and $b_{-,2}$.

The constraints described above leave five free parameters, which correspond to the large-scale perturbations of the model. Three of them are related to the phason strains [2]. In the vicinity of the ground state the corresponding infinitesimal perturbations have the form

$$b_{\pm,i} \to b_{\pm,i} + \epsilon + \mathrm{o}(\epsilon) \qquad i = 1, 2$$

for the phason strain with six-fold symmetry, and

$$b_{-,i} \to b_{-,i} + \epsilon + \mathrm{o}(\epsilon) \qquad b_{+,i} \to b_{+,i} - \epsilon + \mathrm{o}(\epsilon)$$

for the four-fold symmetric phason strain. Another degree of freedom is due to the degeneration of the ground state of the system, and corresponds to simultaneous multiplication of $b_{+,1}, b_{+,2}, b_{-,1}$ and $b_{-,2}$ by the common phase factor $\mathrm{e}^{\mathrm{i}\alpha}$. The last perturbation, which is still consistent with the constraint (8), and is not reduced to the rescaling of z, has the form

$$b_{\pm,1} \to b_{\pm,1}(1 + \epsilon + \mathrm{o}(\epsilon)) \qquad b_{\pm,2} \to b_{\pm,2}(1 - \epsilon + \mathrm{o}(\epsilon)) \tag{9}$$

in the first order in ϵ. We need, however, to keep up to the second order in ϵ, because the expected term is quadratic in the perturbation. More accurate algebra, similar to that used in [2] (see the appendix), gives rise to the following expressions:

$$\begin{aligned}\log(b_{\pm,1}) &= \log(6\sqrt{3}) - \sqrt{3}\log(2+\sqrt{3}) + \epsilon - \epsilon^2/6 + \mathrm{o}(\epsilon^2)\\ \log(b_{\pm,2}) &= \log(6\sqrt{3}) - \sqrt{3}\log(2+\sqrt{3}) - \epsilon - \epsilon^2/6 + \mathrm{o}(\epsilon^2).\end{aligned} \tag{10}$$

The corresponding correction to the entropy per vertex is computed in the same way, and gives

$$\sigma = \sigma_0 - \epsilon^2/3 + \mathrm{o}(\epsilon^2). \tag{11}$$

It remains to find the value of ϵ, which corresponds to the elementary excitation under consideration, that is to a simultaneous increase in the phases Φ_i^+ and Φ_j^- by 2π. In the terms of the integral equation (7) this means that the imaginary part of the integral

$$I = \int_0^{b_{\pm,1}} f_\mp(z)\,\mathrm{d}z \tag{12}$$

is increased by $2\pi/M$ (this integral diverges at $z = 0$, but the imaginary part can be properly regularized). Under the action of perturbation (10) the imaginary part of I is increased by ϵ (see the appendix), which gives rise to the following correction to the entropy per vertex:

$$\sigma_v - \sigma_{v0} = -\frac{4\pi^2}{3M^2}. \tag{13}$$

The corresponding critical exponent x_i from (6) is equal to

$$\frac{4\pi}{3\sqrt{3}} = 2.418\,399\,152\ldots$$

which agrees well with the numerical results (see table 5).

5. Discussion

It is generally believed that the low-lying excitations in the statistical models reflect the macroscopic and universal properties. Their study is of special interest in the case when the *slow variables* describing the system are not explicitly expressed through the microscopic parameters of the model. This is the case for the square–triangle tiling model. The presented results shed new light on the question formulated in [2]: ‘Are six-fold symmetric square–triangle tilings really distinct from all others?’ The peculiar character of six-fold symmetric states (in particular the twelve-fold symmetric one) is supported by the following arguments. First, instead of two types of particle–hole excitation for a non-symmetric state (one for right-moving particles, the other for left-moving ones) there is only one excitation of this sort for the symmetric state. Second, the correction to the entropy density for some excitations of the symmetric state as a function of the system size L contains next-to-leading term L^{-2} a contribution, proportional to $L^{-12/5}$. Both effects are independent of the choice of the representation of the model, which is to say that the six-fold symmetric state belongs to the different universality class.

The spectrum of the transfer matrix does not give the full information on the nature of the excitations. Nevertheless, the way the excitations have been generated gives an indication of the corresponding local operators. As a matter of example, consider the particle–hole excitation in the ideal Fermi liquid

$$|\phi\rangle = \left(\prod_{k_{\min}<k<k_{\max}} a_k^\dagger\right)|0\rangle$$

where $|\phi\rangle$ is the ground state of the model. This state describes, in particular, random tilings of plane by 60°-rhombi [9, 10]. Such excitations can be created by the action of a macroscopic density operator

$$\hat{\rho}_f = \sum_x f(x) a^\dagger(x) a(x)$$

where $f(x)$ is a slow-varying function of a coordinate x. Pursuing the analogy with this model, one might expect that the particle–hole excitations in the square–triangle random tiling are related to the correlation functions of the density of right- and left-moving particles. If this is the case, then the anomalous $L^{-12/5}$ scaling of the correction to the entropy density for two particle–hole excitations may reflect the presence of an irrelevant field with the scaling dimension equal to $\frac{12}{5}$ in the product of two density operators [3].

If the hypothesis of the conformal invariance of the model holds for the ground state (and for other states with six-fold symmetric phason gradient), then the absolute values of the correction to the free-energy density are also to be taken into account. This means, in particular, that there should be a field with the scaling dimension equal to $4\pi/3\sqrt{3}$. Whether or not the model exhibits the conformal symmetry in the thermodynamic limit is, however, still an open question.

Appendix

This appendix is concerned with the derivation of the formulae for the symmetric Umklapp excitation in the thermodynamic limit. First of all, we figure out the value of ϵ in (9), which corresponds to the transfer of one particle from one Fermi point to another. In order to compute the phase shift in (12) it is convenient to use the uniformization, similar to that proposed in [2]

$$z = \frac{b_{\pm,1} t^6 - b_{\pm,2}}{t^6 - 1}. \tag{A1}$$

The poles of the form $f_\pm \mathrm{d}z$ are situated at the points $\exp(\pi \mathrm{i} n/6)$ and $\exp(a + \pi \mathrm{i}(n+1/2)/6)$ on the plane of the complex variable t, where the parameter a is related to ϵ from (9)

$$\epsilon = 3a + \mathrm{o}(a).$$

In the considered case, when $b_{+,(1,2)} = b_{-,(1,2)}$ the form $f_\pm\,\mathrm{d}z$ is rational in the variable t (see [2]). The first derivative of the phase (12) with respect to a is given by

$$\frac{\partial}{\partial a} \int_0^{b_{\pm,1}} f_-(z)\,\mathrm{d}z = 3\mathrm{i}a.$$

The transfer of one particle corresponds to the shift in the imaginary part of (12) by $2\pi/M$, hence the corresponding value of ϵ is equal to

$$\epsilon = \frac{2\pi}{M}.$$

Because the correction to the entropy density is of the second order in ϵ, the values of $b_{\pm,(1,2)}$ have to be computed with the same accuracy. They can be obtained from the following condition

$$\mathrm{Re} \int_\infty^{b_{\pm,(1,2)}} \left(\sqrt{3} f_\pm - \frac{1}{z} \right) \mathrm{d}z = -\log(b_{\pm,(1,2)})$$

which gives rise to formula (10). The expression for the entropy per vertex in the case of the symmetry between right- and left-moving particles can result from the equation

$$\sigma_v - \log(b_{\pm,(1,2)}) = \mathrm{Re} \int_0^{b_{\pm,(1,2)}} \left(f_\pm - \frac{1}{z} \right) \mathrm{d}z.$$

In the second order in ϵ this gives

$$\sigma_v = \sigma_{v0} - \epsilon^2/3 + \mathrm{o}(\epsilon^2)$$

whence follows the formula (13).

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